DEVOLUTION OF THE FISHER EQUATION:
Rational Appreciation to Money Illusion

ABSTRACT

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The "Fisher equation" is one of the best-known equations in economics. What is little known and appreciated, however, is that Fisher authored two forms of this equation. According to the Conventional Fisher Equation (CFE), the nominal interest rate (i) is a linear function of the *ex ante* real interest rate (r) and expected inflation (π):

$$i = r + \pi + r\pi \tag{1}$$

Expected inflation (expected appreciation of goods) is defined as the percent change in the price of goods (P): $\pi \equiv (EP_{t+1} - P_t)/ P_t$. For "small" values of expected inflation, the CFE embodies the "Fisher effect" of a (near) one-to-one relationship between expected inflation (π) and the nominal interest rate (i).

The CFE is one of the most widely cited and employed equations in economics. A careful reading of Fisher's extensive works, however, leads to three surprising conclusions: (1) Irving Fisher never published an *explicit* form of the CFE. (2) The Original Fisher Equation (OFE, 1896), which was written in terms of the *expected* appreciation of money, is a separate equation and not a simple transposition of the CFE or reformulation of its temporal base. (3) Fisher's (1930) substitution of goods appreciation (π) for expected money appreciation (a) in his empirical equations resulted from his conviction that market expectations are more accurately characterized by "money illusion" than by rational behavior.

The paper traces the evolution of Fisher's theory from its original formulation in terms of rational appreciation to its reconstruction under money illusion. The objective is to recover the OFE and explain its distinction from and displacement by the CFE. The OFE reflects Fisher's early interest in mathematical economics and rational behavior. It was derived under the assumption of rational foresight and measurement. In his later work, Fisher (1930) introduced an *implicit* form of the CFE that substituted the appreciation of goods (π) for the

expected appreciation of money (a). This change of emphasis resulted from Fisher's growing skepticism about the rationality of market expectations and measurements. It assumed that market participants make decisions based on money illusion. The modern form of the CFE is a curious hybrid of Fisher's old and new thinking. It assumes farsighted behavior, but it adopts the definition of "appreciation" that Fisher and modern financial economists associate with non-rational behavior.

What did Fisher mean by "money appreciation"? In general, he meant a rise in the value of a monetary asset relative to another asset or good. Appreciation under a modern fiduciary standard refers to the rate of change in the value of (paper) money expressed in terms of commodities. A modern name for this concept is the real return on money (Eden, 1976). If the value of goods is P, then the value of one unit of money (v) is 1/P. Where Fisher used "expected appreciation of money," modern economists usually substitute "expected deflation." Adding to the confusion is the fact that the concepts are interchangeable in the special case where current and future prices are known with perfect certainty and where rates of change are calculated with reference to a common base. In the case of uncertainty (where expectations must be formed over uncertain future values), the mathematical definitions are *not* equivalent. The non-equivalence of the two specifications in the uncertainty case is true even if calculations of the *expected* rate of change use a common base. This non-equivalence is a consequence of Jensen's inequality.

Fisher, as he readily acknowledged, was not the first person to advance the theory that the interest rate adjusts to changes in money value. Humphrey (1983) traces the lengthy development of this idea. Fisher's contribution was being the first to write an equation for the relationship (Humphrey, 1983; Dimand, 1999a). Fisher was also the first to clearly derive the equation. His original equation, however, is *not* the one that is commonly attributed to him.

This is fortuitous in the sense that the CFE is a misspecification of the relationship between nominal and real yields when market participants form *rational* expectations over *uncertain* future prices (Eden, 1975, 1976; Blejer and Eden, 1979; Kochin, 1980; and Benninga and Protopapadakis, 1983). But what if market expectations are based on money illusion rather than rational behavior? What equation is appropriate then? Ironically, it was Fisher's (1930) rejection of the rationality postulate that led him to an implicit form of the conventional specification.

Despite frequent assertions to the contrary, the original source of the "Fisher equation" is not *The Theory of Interest*. To uncover the nature of the theoretical relationship between *ex ante* real and nominal interest rates, we must take Fisher's (1930, p. 39) advice and consult *Appreciation and Interest*.

The original Fisher equation (OFE), when applied to a world of paper money ("money") and commodity money ("commodities"), is expressed in terms of the *expected* appreciation of money (a) and written with the commodity (real) interest rate (j), Fisher's "virtual interest in commodities," as the left-hand variable. That is,

$$j = i + a + ia \tag{2}$$

In this form, the original equation can easily be misinterpreted as the CFE. The CFE, however, is not the same equation as the OFE for two reasons: (1) the interactive terms are different ($r\pi \neq - ia$) and, more importantly, (2) the measures of *expected* appreciation are different ($\pi \neq - a$). The inequalities occur even in the perfect certainty case due to differences in the base period used for calculating discrete changes.

Historians of thought, while employing Fisher's (1896) terminology, have inadvertently contributed to the misunderstanding of Fisher's theory. Tobin (1997, p. 374), Howitt (1992, 2, p. 123), and Dimand and Geanakoplos (2005), for example, describe the

Fisher equation using the conventional specification (1). Dimand (1999a) accurately reproduces the original Fisher equation (5) and points out that Fisher's money and commodities model used the "(expected) purchasing power of money." In defining "expected inflation as the difference between real and nominal interest rates," however, Dimand (1997, p. 442; 1999a, p. 748; 1999b, p. 36) assumes that there is no difference between the OFE and the CFE. Humphrey (1983), in an otherwise illuminating discussion, uses the conventional, rather than the original, Fisher equation in describing Fisher's contribution to the history of thought. The mistake is in viewing the two equations as equivalent. The CFE is a rival equation not a transposition of the OFE.

Although it would not surprise Irving Fisher (see below), the continuing popularity of the CFE in the conventional economics literature is puzzling. It is a well-established proposition in the finance literature that the CFE provides a biased estimate of the relationship between nominal and real bond yields when expectations are formed rationally over uncertain future prices (Benniga and Protopapadakis, 1983; Blejer and Eden, 1979; and Kochin, 1980). What is the extent of the bias? If the size of the bias is small, the CFE may be used as a reasonable approximation to the OFE. Theoretical models suggest that this is a risky assumption, particularly in cases where price level volatility is large (Eden, 1975, 1976; Sarte, 1998), expectation horizons are long (McCulloch and Kochin, 2000), or individual expectations are diffuse (Kochin, 1980). Fama (1975, 1976), in an apparent attempt to avoid inflation-uncertainty bias, wrote the Fisher relationship in terms of the expected value of money [E(1/P)]. In doing so, he inadvertently rediscovered the OFE.

Why did Fisher (1896) insist on formulating the problem in terms of the expected appreciation of money (a) rather than expected deflation of commodities (π)? One cannot be sure, but one should not overlook a simple explanation. As a neoclassical economist, Fisher

defined the value of money as the inverse of the price of goods. It verges on the obvious that if you use the wrong definition for the price of money, you will end up with a biased measure of the expected appreciation of money.

Jensen did not publish his formal proof concerning convex functions until 1906. Did Fisher (1896) have an understanding, intuitive or otherwise, of Jensen's inequality? One cannot say with absolute certainty, but a careful reading of his subsequent work on index numbers suggests he did. Schumpeter (1954, pp. 1091) points out that the work of Fisher and others on index numbers was the "statistical complement" to the "theoretical discussion on the purchasing power of money." Fisher knew that the choice of definition for money appreciation has important consequences. As "the greatest expert of all time on index numbers" (Tobin, 1987, p. 369), Fisher understood that care must be exercised in the calculation of mean values.

A special case of Jensen's inequality is the difference between arithmetic and harmonic means. Fisher knew and often used the fact that the arithmetic mean [A(P)] of a variable P is greater than (or equal to) its harmonic mean [H(P)]. The harmonic mean, H(P), is equal to one over the arithmetic mean of the variable's inverse: $H(P) = 1/[A(1/P)$. That is, $A(P) \geq H(P) = 1/[A(1/P)]$. This implies: $A(1/P) \geq 1/A(P)$. In this special case, all prices (P) are known with certainty and given equal weights (1/n). The generalization to the uncertainty case is straight-forward. If P is a discrete random variable, then the mathematical expectation of P, E(P), is a weighted average of individual prices with the weights of individual prices being probabilities that sum to one. Replacing the arithmetic operator (A) with the expectation operator (E), results in the inequality of current interest: $E(1/P) \geq 1/E(P)$.

Fisher's choice of terminology is further evidence of his sophisticated understanding. When explaining the *theoretical* connection between nominal and real interest rates, Fisher consistently used terms such as "expected change in the value of money," "expected

appreciation of money," or "expected change in the purchasing power of money." Fisher (1896, 1905, and 1930) used the same terminology in all of his major works on the theory of interest. Fisher bemoaned, over and over again, the apparent inability of people to grasp the concept of money value. He did not take it to be a matter of inconsequence that people found it easier to calculate in terms of prices than money values. Indeed, Fisher insisted that his weekly Index Number of Wholesale Prices be published as the *inverse* of the original price series.

The strongest argument for Fisher's precocity lies in his empirical studies. In his extensive empirical investigations, reported in detail in 1896, appreciation was consistently calculated as the *expected* change in the *reciprocal* of the price level. It was the OFE, not the CFE, that Fisher chose when he processed the data from the uncertain world.

The empirical question Fisher (1896) attempted to address was the extent to which *ex post* appreciation (a*) was captured by *expected* appreciation (a). In testing his theory, Fisher (1896) used a variety of alternative definitions of money X and asset/good Y: gold and wheat (ch. II), gold and paper (ch. VIII), gold and silver (ch. IX), and money and commodities (ch. X). It is not until part II that he introduces the modern convention of using fiduciary money and (aggregate) commodities as the two standards. It is in part II that Fisher drops the simplifying assumption of perfect foresight and makes clear that the OFE is in terms of *expected* appreciation. Fisher uses bond market and price data from seven countries to examine the extent to which market interest rates adjust to the "appreciation of money in commodities." Fisher's examples reflect the period of investigation: money is the (relatively) appreciating standard and (aggregate) commodity is the (relatively) depreciating standard. Money appreciates when commodity prices (P) go down and depreciates when prices go up.

In part II, Fisher (1896, p. 43) uses the equation derived in part I to obtain an empirical measure of "expected appreciation" of money from market data. He achieved this remarkable

feat by exploiting the difference in the yields of commodity (gold coin) bonds and paper (currency) bonds. The "realized" (i.e. market determined) yields on the commodity and on paper were calculated independently as "the rate of interest which will render the 'present value', at the date of purchase, of all the future benefits to January, 1879, equal to the purchase price" (p. 42, n. 4). Using the resulting paper yield (i) and the commodity yield (j), Fisher employed the OFE to solve for the *expected* appreciation of money (a); that is, "that rate of appreciation which would have made the two interest rates equally profitable" (p. 43, n. 4). He compared this *forward-looking* measure of expected appreciation with the realized *(ex post*) appreciation of money and discovered that expected (*ex ante*) appreciation (a) consistently *under*-predicted actual appreciation (a*).

Of Irving Fisher's works on interest rate behavior, the one that is most frequently cited is *The Theory of Interest*. Although one will not find an explicit representation of the CFE in any of Fisher's works, it is easy to see how a reader who consults only *The Theory of Interest* would find support for the conventional interpretation. Setting a pattern for subsequent research, Fisher (1930, ch. XIX) examined the correlation between the nominal interest rate and the rate of change in the price of commodities. As is well known, he found a weak contemporaneous correlation between the rate of change of commodity prices and the nominal interest rate. Applying a distributed lag model of his own invention (1925, 1938), Fisher found that *past* inflation influenced both long-term and short-term interest rates with a long and variable lag.

Fisher's *empirical* model is the source of confusion over the Fisher equation. Why did Fisher switch from expected money appreciation to lagged inflation in his post-1896 empirical work? One cannot be absolutely sure, but the change in emphasis is dramatic in light of Fisher's previous insistence on using money value. The most likely answer can be found in

Fisher's psychological theory of expectations. Fisher's early empirical work (1896) led him to question the rationality of market expectations. Conversations with businessmen and workers further convinced him that the value of money was too subtle a concept for ordinary people to comprehend. The popular view, according to Fisher (1930, p. 399) is that "money itself does not change." If this is the case, then bond market participants do not form expectations over the value of money and a viable empirical model of the interest rate cannot be conditioned on the OFE.

In 1896, Fisher had yet to introduce the concept of "money illusion." Fisher (1896, p. 11) made clear that he was "regarding money as a standard of value and not as a medium of exchange." Money is a measure of value just as a yard is a measure of length. Contracts, whether expressed in money or yards, should be adjusted to take proper account of changes in the units of measurement. Fisher derived the OFE under the assumption of rational measurement and rational expectations, but his early empirical work suggested that interest rates fell significantly short of anticipating subsequent money appreciation.

As early as 1896, Fisher was beginning to have second thoughts about rational behavior. In spite growing doubts, Fisher (1896) left open the possibility that inadequate interest rate adjustment might be due to "imperfection of foresight." Anticipating the regime switching literature (Barsky, 1987), Fisher recognized that the mere possibility of a monetary regime change would provide rational grounds for such imperfection (Fisher, 1896, chapter VIII). At the end of his career, Fisher (1946) admitted his reluctance to shed the rationality assumption. Fisher's own illusion about educating the masses on the value of the dollar was gradually undermined by his empirical studies and his business dealings.

Money illusion is traditionally defined as a situation where market participants make economic decisions based on money prices rather than theoretically correct relative prices and

real wealth (Patinkin, 1965, pp. 22-23). Money illusion, in this sense, is a violation of the "homogeneity postulate" (Leontief, 1936). Workers suffering from money illusion bargain in terms of money wages rather than real wages. Business managers, to the extent they suffer from the disease, fail to adequately take account of the general price level in making pricing and output decisions. Fisher used such notions throughout his collected works, especially in *The Money Illusion* and other business cycle writings where some type of fooling assumption is required to explain output and employment effects of monetary disturbances.

Money illusion, according to Fisher (1928, p. 4) is "the failure to perceive that the dollar, or any other unit of money, expands or shrinks in value." Money illusion results in an incorrect measure of the *change* in the appreciation of money (the growth of the king's girdle). The Patinkin form of money illusion is an extreme case when money value (the yardstick measurement) is perceived not to change in value at all; when a yard is a yard and "a dollar is a dollar" (Fisher, 1896, p. 35; 1930, p. 399).

The presence of money illusion limits the *direct* impact of expected appreciation on interest rates. The possibility remains of a roundabout influence. In various writings, Fisher conjectured that changes in commodity inflation would have an *indirect* and *lagged* impact on the nominal interest rate and other variables. According to Fisher, price changes may have an impact on interest rates even in the presence of imperfect foresight and money illusion. Sluggish price changes and the resulting trade fluctuations put indirect pressure on loan markets and interest rates. To capture the lagged effect of prices on interest rates, Fisher (1925, 1930) developed the distributed lag model.

Fisher's concept of money illusion did not rule out the possibility of imperfect foresight with respect to the price of goods. Based on empirical observation, Fisher came to believe that market participants exhibit complex psychological behavior: both foresight and illusion

influence market outcomes. Fisher would not be surprised by modern psychological studies that find inconsistencies and inaccuracies in people's calculation of money values (cf. Safir, Diamond, and Tversky, 1997; Fehr and Tyran, 2001). The interaction between money illusion and imperfect foresight provided Fisher a rationale for replacing expected appreciation with lagged inflation.

Friedman and Schwartz (1982, p. 547) note that Fisher's 1930 empirical work has "less economics" than his earlier works (1896, 1907). It is stretching matters, however, to attribute the loss of economics to his adoption of the adaptive expectations hypothesis. The concept of adaptive expectations is an interpretation superimposed on Fisher's (1930) lagged adjustment model by subsequent researchers. What constitutes the "loss" is the switch in emphasis from market rationality to market psychology. Fisher's empirical model could not assume full rationality if market psychology was dominated by widespread money illusion. If Fisher's theory of inflation psychology is correct, then empirical studies using a backward-looking specification should more accurately predict the behavior of the nominal interest rate than those based on a forward-looking specification.

We are now in a position to understand why Fisher called appreciation a "subtle conception." The debate over the specification of the Fisher equation involves two subtle issues of measurement that Fisher never adequately disentangled. To return to Fisher's colorful analogy, measurement problems arise if the length of the yardstick depends on the size of the king's girdle. One measurement problem results if people suffer from girdle illusion (Patinkin-style money illusion); that is, they fail to adjust the yardstick with the changing size of the king's girdle. Realization that the yardstick changes, however, is not enough to eliminate measurement issues. A second measurement problem occurs if one uses an improper yardstick (the Jensen inequality problem) in calculating the actual and expected rates of change in the

king's girth (the expected rate of change in money value). It matters, for example, whether the yardstick is made of wood (linear measurement) or cloth (convex measurement). The point of reference on the king's body (base period problem) is also important for such calculations.

The implicit form of the CFE used by Fisher assumed both money illusion and improper measurement of money value. By the early 1970s Fisher's (1930) distributed lag model of inflation was commonly interpreted as a form of the adaptive expectations hypothesis. The modern specification of the CFE has restored some semblance of rationality by emphasizing forward-looking forecasts of goods prices. Although it is now common to superimpose the rational expectations hypothesis on the CFE, Fisher would not be satisfied. To Fisher, rational appreciation requires both proper measurement of expected money value and unbiased expectations. Full rationality requires a return to the 1896 vision of Fisher. This, in fact, was the course taken by Fama (1975, 1976).