【Notes and Communications】

Cournot's Limit Theorem and the Cost Functions: A Diagrammatic Exposition

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I Introduction

Cournot's ([1838] 1960, 90) assertion that when the number of producers in the market is sufficiently large the Cournot equilibrium price is equalized to the marginal cost of each producer is known as Cournot's limit theorem.[1] Cournot (ibid., 91–92) suggested that the theorem holds with increasing marginal costs and without fixed costs, while the theorem does not hold with decreasing marginal costs, though he did not provide formal proof of his suggestions.

Thereafter, the validity of the limit theorem has been examined under various settings. Frank (1965) examined the theorem in the Cournot model admitting heterogeneous firms.[2] Using the Cournot model with identical firms and without fixed costs, Ruffin (1971) showed that the theorem holds.[3] Using an example due to Novshek and Sonnenschein (1978), Tirole (1988, 227) demonstrated that with U-shaped average cost curves and without fixed costs, the Cournot equilibrium converges to the perfectly competitive equilibrium when the output minimizing the average cost converges to zero. Martin (2002, 39) pointed out that the essential requirement for the Cournot equilibrium to converge to the perfectly competitive equilibrium as the number of firms becomes large is that there is no initial downward-sloping segment of the average cost curve.

This note provides a diagrammatic method to examine the validity of Cournot's limit theorem under the various types of cost functions, as described above. Our method is an extension of Negishi's (2001, 92–96) method, where

1 Negishi (1989, 244).

2 For the effects of the number of firms on the equilibrium price of the Cournot model admitting heterogeneous firms, see Okuguchi (1973) and Uchiyama (2018).

3 For the effects of the number of firms on the equilibrium price of the Cournot model with identical firms, see Seade (1980) and Amir and Lambson (2000).

the Cournot duopoly with identical producers and linear demand and cost functions is illustrated on the price–quantity plane.[4] Using this method, it is confirmed that Cournot's suggestions are correct.

The rest of the note is organized as follows. Section II provides the diagrammatic method to examine Cournot's suggestions and applies the aforementioned method to examine the limit theorem under increasing marginal costs without fixed costs. Section III applies the method to examine the limit theorem under decreasing marginal costs. Section IV concludes this note.

II　Diagrammatic Exposition of the Limit Theorem

In this section, we provide the diagrammatic method to demonstrate the Cournot equilibrium among $n(>1)$ producers on the price–quantity plane. Furthermore, we apply the method to show that Cournot's limit theorem holds with increasing marginal costs and without fixed costs.

Before utilizing the diagrammatic method, we review Cournot's argument about the limit theorem. As for the theorem, he describes it as follows:

> 50.　The effects of competition have reached their limit, when each of the partial productions D_k is inappreciable, not only with reference to the total production $D = F(p)$, but also with reference to the derivative $F'(p)$, so that the partial production D_k could be subtracted from D without any appreciable variation resulting in the price of the commodity. This hypothesis is the one which is realized, in social economy, for a multitude of products, and, among them, for the most important products. It introduces a great simplification into calculations, and this chapter is meant to develop the consequences of it.
>
> According to this hypothesis, in the equation
>
> $$D_k + [p - \phi_k'(D_k)] \cdot \frac{dD}{dp} = 0,$$
>
> the term D_k can be neglected without sensible error, which reduces the equation to
>
> $$p - \phi_k'(D_k) = 0.$$
>
> （Cournot［1838］1960, 90）

In the quotation, D_k, D, and $F(p)$ are the output of producer k, total output, and the demand function, respectively, while p and $\phi_k(D_k)$ are the price of the com-

4　For the other types of the diagrammatic methods to demonstrate the Cournot oligopoly on the price–quantity plane, see Ikema（1991）and Ishikawa（1997）.

modity and the cost function of producer k, respectively. $\phi_k'(D_k)$ represents the derivative of $\phi_k(D_k)$ with respect to D_k, i.e., the marginal cost. Note that $D_k + [p - \phi_k'(D_k)] \cdot dD/dp = 0$ in the quotation stands for the first order condition for profit maximization by producer k under the Cournot competition.

Thus, Cournot insists that, "when each of the partial productions D_k is inappreciable [. . .] with reference to the total production $D=F(p)$," i.e., the number of the producers in the market is sufficiently large, "[t]he effects of competition have reached their limit" and the market price under the Cournot competition, p, is equalized to the marginal cost, $\phi_k'(D_k)$. This assertion is his limit theorem.

Moreover, as for the cost functions and the limit theorem holding, Cournot describes this as follows:

> In the hypothesis under consideration, all the functions $\phi_k'(D_k)$ must be considered to increase with D_k. Otherwise the gross value of the product
> $$pD_k = D_k \cdot \phi_k'(D_k)$$
> would be less than the costs of production, which are
> $$\phi_k(D_k) = \int_0^{D_k} \phi_k'(D_k)\, dD_k.$$
> (Cournot [1838] 1960, 91)

Note that, in the quotation, from

$$\phi_k(D_k) = \int_0^{D_k} \phi_k'(D_k)\, dD_k \quad (= \phi_k(D_k) - \phi_k(0)),$$

$\phi_k(0)=0$ holds, i.e., there are no fixed costs. Therefore, Cournot suggests that the limit theorem holds with increasing marginal costs and without fixed costs.

Now we provide the diagrammatic method to examine the above argument by Cournot. First, we demonstrate the Cournot equilibrium among n (>1) producers on the price–quantity plane. Throughout this note, we assume that there are n identical rival producers in the market and that the market demand curve is indicated by straight line AB with the slope $-a<0$ on the price–quantity plane.[5] In this section, we assume increasing marginal costs without fixed costs. In Figure 1, the marginal and average cost curves of each producer are indicated by curves CD and CM, respectively. Throughout this note, the slopes of straight

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5 Examples of identical producers or plants are employed by Cournot: "Let us now imagine two proprietors and two springs of which the quality are identical, and which, on account of their similar positions, supply the same market in competition" (Cournot [1838] 1960, 79). "If there were 3, 4, ⋯, n producers in competition, all their conditions being the same, equation (3) would be successively replaced by the following:" (Cournot [1838] 1960, 84).

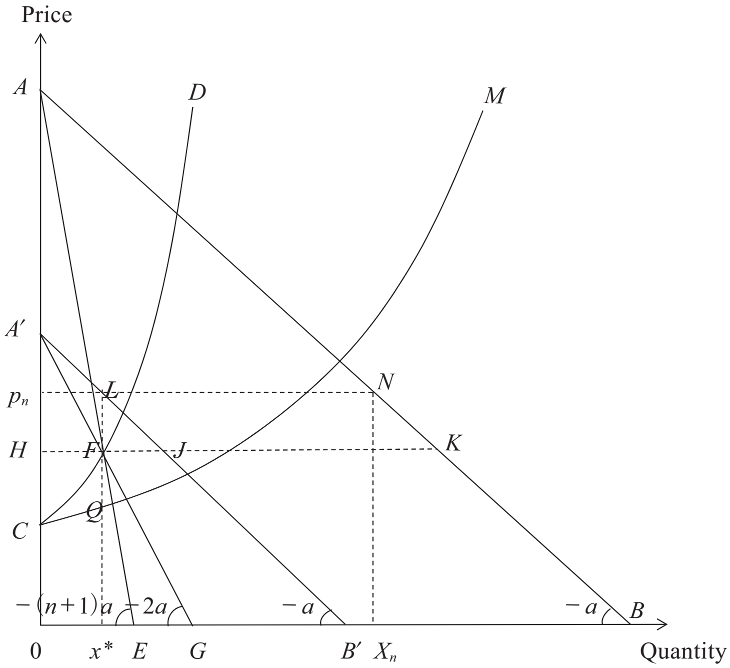

Figure 1　The Cournot Equilibrium and the Limit Theorem

lines AE, $A'G$, and $A'B'$ are $-(n+1)a$, $-2a$, and $-a$, respectively. Note that $HF = FJ$, $JK = LN = (n-1) \times HF$, and $p_nN = n \times HF$ hold.[6]

The Cournot equilibrium among n producers is given by point N. The equilibrium price and total output are p_n and X_n (= $n \times HF$), respectively, while the output of each producer is x^* (= HF). We can confirm that this situation represents the Cournot equilibrium as follows. If each producer chooses x^*, then, under the Cournot competition, he or she observes that the total output of his or her rival producers is $(n-1)x^*$ (=JK=LN). The producer's residual demand curve and marginal revenue curve are represented by $A'B'$ and $A'G$, respectively. The producer's output maximizing his or her profit is x^*, given by point F (the intersection point of the marginal revenue curve $A'G$ and the marginal cost curve CD). Hence, once each producer chooses x^*, any producer does not want to change x^*. Therefore, the situation where each producer chooses x^* is the Cournot equilibrium.[7]

Second, we confirm that the limit theorem holds. In Figure 1, each produc-

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6 From the slopes of $A'G$ and $A'B'$, $A'H/HF = 2a$ and $A'H/HJ = a$ hold, so that $HJ/2 = HF$ =VFJ is obtained. From the slopes of AF and AK, $AH/HF = (n+1)a$ and $AH/HK = a$ hold, so that $HK = (n+1) \times HF$ is obtained. Thus, $JK = HK - HF - FJ = (n-1) \times HF$ holds.

7 Note that, with increasing marginal costs and the negative slope of the market demand curve, as in Figure 1, the stability condition introduced by Hahn (1962) is satisfied.

er obtains positive profit in the Cournot equilibrium, which is represented by $p_n > Qx^*$. This positive profit attracts new entrants, so that n becomes larger. The divergence between the market price and the marginal cost in the Cournot equilibrium is depicted by LF. From the slope of LJ and $HF = FJ$, $LF/FJ = LF/HF = a$ holds, so that $LF = a \times HF$ is obtained. As n becomes larger, the slope of AE becomes steeper and, point F approaches point C along CD and, therefore, HF contracts. Thus, as n becomes larger, LF contracts. Since each producer obtains positive profit for any n, n continues to increase. Considering that n becomes sufficiently large, point E converges to point 0. Then, HF and LF converge to zero, which implies that the Cournot equilibrium price converges to the marginal cost of each producer. Hence, as Cournot suggests, the limit theorem holds with increasing marginal costs and without fixed costs.

III The Limit Theorem and Decreasing Marginal Costs

In this section, we diagrammatically show that the limit theorem does not hold with decreasing marginal costs. Cournot suggests such possibility:

> Thus, wherever there is a return on property, or a rent payable for a plant of which the operation involves expenses of such a kind that the function $\phi_k'(D_k)$ is a decreasing one, it proves that the effect of monopoly is not wholly extinct, or that competition is not so great but that the variation of the amount produced by each individual producer affects the total production of the article, and its price, to a perceptible extent.
>
> (Cournot [1838] 1960, 91–92)

Thus, Cournot suggests that with decreasing marginal cost, "competition is not so great," so that the limit theorem does not hold.

Figure 2 illustrates that there is an initial downward-sloping segment of the marginal cost curve, including the U-shaped marginal cost curve as a special case.[8] The marginal and average cost curves of each producer are depicted by curves MC and AC, respectively. We assume that there is $A'B'$, which is tangent to AC. Let n^* denote n satisfying $-(n+1)a=$ the slope of AE^*, which is a

8 Cournot admits the case of the U-shaped marginal cost curve: "For what are properly called manufactured articles, it is generally the case that the cost becomes proportionally less as production increases, or, in other words, when D increases $\phi'(D)$ is a decreasing function. [. . .] It may happen, however, even in exploiting products of this nature, that when the exploitation is carried beyond certain limits, it induces higher prices for raw materials and labour, to the point where $\phi'(D)$ again begins to increase with D" (Cournot [1838] 1960, 59–60).

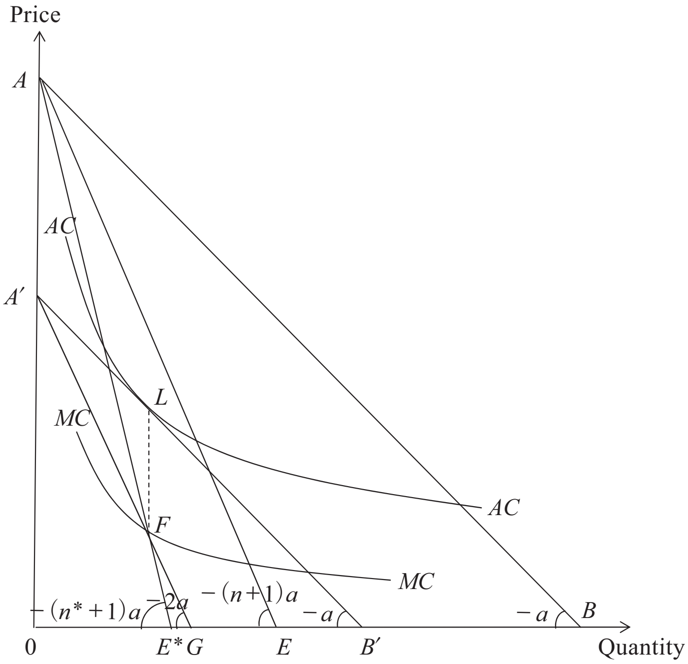

Figure 2　The Limit Theorem and Decreasing Marginal Costs

straight line through point F (the intersection point of $A'G$ and MC). We assume $n^* > 1$ and denote the greatest integer less than or equal to n^* by $\bar{n}$. If $A'B'$ is the residual demand curve of each producer, then $A'G$ is the producer's marginal revenue curve and, on $A'B'$, only point L obtains non-negative profit (maximum profit), hence point L must be vertically above point F.

Since the residual demand curve derived from AE under $n \leq \bar{n}$ is not below $A'B'$, the point obtaining the non-negative profit exists on such residual demand curve. Therefore, the Cournot equilibrium can be found among n producers with $n \leq \bar{n}$ and non-negative profits, represented by the point obtaining maximum profit on such residual demand curve.

On the other hand, the Cournot equilibrium cannot be found among n producers with $n \geq \bar{n}+1$ and non-negative profits. If there is such equilibrium, then the point representing such equilibrium must be on the residual demand curve derived from AE under $n > n^*$. Such residual demand curve is below $A'B'$ and, therefore, below AC, which is a contradiction to the Cournot equilibrium with non-negative profits.

9 Note that, if there are fixed costs, then the theorem does not hold even with increasing or constant marginal costs since there is an initial downward-sloping segment of AC, and in such segment, AC is above MC, as shown in Figure 2.

Thus, even if the Cournot equilibrium among $\bar{n}$ producers with non-negative profits holds, no new producer enters the market since the Cournot equilibrium with non-negative profits does not exist when he or she newly enters the market. Therefore, the number of producers in the market does not exceed $\bar{n}$. This implies that the divergence between the market price and the marginal cost of each producer in the Cournot equilibrium does not become smaller than *LF* in Figure 2, since such divergence = $a \times$ the distance from the vertical axis to the intersection point of *MC* and *AE*. Thus, as Cournot suggests, the limit theorem does not hold with decreasing marginal costs.[9]

IV Conclusion

This note has provided the diagrammatic method to examine the Cournot oligopoly with n (>1) identical producers and non-linear cost functions. Using this method, it has been shown that Cournot's limit theorem holds with increasing marginal costs and without fixed costs, while the theorem does not hold with decreasing marginal costs. Thus, it has been confirmed that Cournot's suggestions are correct.

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